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ASYMPTOTIC ANALYSIS OF PROBLEMS ON THE FREE VIBRATION OF RECTANGULAR TRANSVERSELY ISOTROPIC AND THREE-LAYER PLATES

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This article examines problems concerning the free vibration of transverse isotropic and three-layer rectangular plates (refined theory of bending accounting for shear through the thickness). The problems are described by a system of two equations, the first being of the order 2m (m = 2, 3 for transversely isotropic and three-layer plates, respectively) and the second a singularly perturbed second-order equation containing the small parameter ε. For transversely isotropic plates, ε characterizes the effect of transverse shears, while it characterizes the shear stiffness of the three-layer sandwich in the case of three-layer plates. We construct asymptotic expansions of the solutions with allowance for angular boundary-layer solutions, when the parameter ε is small. In this case, the second equation is a perturbation equation whose solution is in the nature of a boundary layer (edge effect).

Different types of boundary conditions are examined for the initial systems. We study the relationship between the boundary conditions of the initial and truncated problems (with the perturbation equation omitted). Substantiation is provided for the transition from the boundary conditions in the refined formulation to the classical formulation in the neighborhood of points of inflection (i.e., for a piecewise-smooth contour). Use of the Kirchhoff transform is validated for a free edge near a corner. Although a separation of variables is often possible for truncated problems, the complete system of equations does not permit such separation.

In the classical theory of the bending of plates, there is a contradiction between the overall order of the system of equations (two biharmonic equations for the normal deflection and the stream function) and five natural static boundary conditions. Thus, on the free edge, the bending and turning moments, the shearing force, and two forces in the plane of the plate are equal to zero. In the classical theory, four rather than five boundary conditions are established for the free edge if the Kirchhoff transform is used. There are theories which are refinements of the classical theory and make use of more general hypotheses in deriving the equations (allowance for shear through the plate thickness). The contradiction between the overall order of the system and the natural static boundary ocnditions disappears in these theories. The form of the system which is simplest for analytical purposes is probably that presented in [1, 2]. The order of this system is higher than in the classical theory due to the presence of a second-order equation having a solution of the edge-effect (boundary-layer) type.

Researchers have developed a method of changing over from the boundary conditions of the refined theory to the boundary conditions of the classical theory [3-5] (an example

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being the use of the Kirchhoff transform for a free edge) in the case of smooth contours. In the case of rectangular plates (the case of a piecewise-smooth contour), it is necessary to substantiate these transformations near corner points. We propose an approach in which, with allowance for boundary-layer solutions [6, 7], asymptotic analysis is employed to formulate the boundary conditions of truncated problems by the elimination method [8, 9].

1. We will examine a system of two linear equations with constant coefficients in a plane rectangular region $D = \{(x, y): 0 < x < 1, 0 < y < b/a\}$ with the boundary $\Gamma = \bigcup_{l=1}^{4} \Gamma_l$ (Γ_l are sides of the rectangle, numbered in the counterclockwise direction beginning with the side $x = 0$):

$$L_0 w(x, y) = \omega^2 M_0 w(x, y), \quad \varepsilon^2 \Delta v(x, y) = v(x, y). \tag{1.1}$$

Here, L_0 and M_0 are uniformly elliptic operators of order 2m, 2k (m > k), respectively; Δ is the Laplace operator; ω is the natural frequency of the vibrations.

The boundary conditions for system (1.1) have the form

$$B_k w + \sum_{|\alpha|=0}^{N_k} a_{k\alpha}(\varepsilon) D^\alpha v|_\Gamma = 0 \quad (k = 1, 2, 3, \ldots, m, m+1), \tag{1.2}$$

where $\alpha = (\alpha_1, \alpha_2)$; $|\alpha| = \alpha_1 + \alpha_2$; $D^\alpha = \partial^{|\alpha|}/\partial n^{\alpha_1}\partial s^{\alpha_2}$; the order of the operators B_k is equal to m_k ($m_k \leqslant 2m$).

With specific operators L_0, M_0, and B_k and coefficients $a_{k\alpha}$, Eqs. (1.1)-(1.2) describe problems on the free vibration of transversely isotropic and three-layer plates.

We will assume that boundary conditions (1.2) are described in canonical form if they satisfy the following requirements.

A. Let $a_{k\alpha}(\varepsilon) \sim O(\varepsilon^{p_{k\alpha}})$ ($p_{k\alpha}$ are integers). Then quantity $q_{k\ell}$, determined from the relation $q_{k\alpha} = \alpha_1 - p_{k\alpha}$, will be referred to as the ε-th order of the effect on Γ of the operator $a_{k\alpha}(\varepsilon)D^\alpha$ with fixed α. We shall call the number $q_k = \max_\alpha q_{k\alpha}$ the ε-th order of the effect of the k-th boundary condition (1.2) on Γ. We assume that the boundary conditions are written so that $q_k \leqslant q_{k+1}$ (k = 1, 2,...,m).

We will use T_k^0 to represent a differential operator whose terms in the k-th boundary condition have an ε-th-order effect equal to q_k and we will call it the principal part of the boundary conditions in v. We will use T_k to represent the remaining terms. Then conditions (1.2) have the form

$$B_k w + T_k^0 v + T_k v|_\Gamma = 0.$$

B. If in the last s conditions (1.2) $q_{m-s+1} = \ldots = q_m = q_{m+1}$, then the last condition in (1.2) k = m + 1 contains the maximum derivative with respect to the normal to the boundary in the principal part.

It is fairly often the case in practical problems that ε is small. It is known that the solution of the perturbation equation is in the nature of a boundary layer and is nontrivial only in the neighborhood of the boundary Γ; this solution rapidly approaches zero with increasing distance from the boundary.

Let us proceed to the formulation of the degenerate problem at $\varepsilon \to 0$. It is obviously connected with the solution of the first equation of system (1.1) — for which we must formulate m boundary conditions on Γ.

The algebraic approach [10, 11] is based on the fact that the last condition in (1.2) is dropped and the function v is omitted from the remaining conditions. We thus obtain the boundary conditions of the degenerate problem

$$L_0 w_0(x, y) = \omega^2 M_0 w_0(x, y), \; B_k w_0|_\Gamma = 0 \quad (k = 1, 2, 3, \ldots, m). \tag{1.3}$$

As is known, in the general case (such as a free edge), the boundary conditions of degenerate problem (1.3) do not coincide with the boundary conditions of simplified plate theories. The difficulties encountered in formulating the boundary conditions of degenerate problems in such cases are related to the fact that complete information is needed on the structure of the solution of the perturbation equation.

Use of the elimination method [8, 9] developed for plates and shells with a smooth contour [4, 5] and in the neighborhood of points of inflection [7] makes it possible to overcome these obstacles. This method is based on the fact that information on the solution of the perturbation equation is accounted for in the first m boundary conditions (1.2).

We will assume (condition A) that inhomogeneous equations (1.1) with inhomogeneous boundary conditions (1.2) have a unique solution if the solvability condition is satisfied.

We will construct the solution of problem (1.1)-(1.2) in the following form ($\varepsilon \ll 1$) [6]

$$
\begin{gathered}
w_\varepsilon(x, y) = \sum_{i=0}^{n} \varepsilon^i w_i(x, y) + \varepsilon^{n+1} Z_n(x, y), \\
v_\varepsilon(x, y) = \varepsilon^\beta \Big\{ \sum_{i=0}^{n} \varepsilon^i [Q_{1i}(\xi_1, y) + Q_{2i}(x, \eta_1) + Q_{3i}(\xi_2, y) + Q_{4i}(x, \eta_2) + \\
+ \varepsilon^\gamma (R_{1i}(\xi_1, \eta_1) + R_{2i}(\xi_2, \eta_1) + R_{3i}(\xi_2, \eta_2) + R_{4i}(\xi_1, \eta_2))] + \varepsilon^{n+1} z_n(x, y) \Big\},
\end{gathered}
\tag{1.4}
$$

where $\xi_1 = x/\varepsilon$; $\xi_2 = (1 - x)/\varepsilon$; $\eta_1 = y/\varepsilon$; $\eta_2 = (a - y)/\varepsilon$; $Z_n(x, y)$ and $z_n(x, y)$ are the remainders of the expansion. It should be noted that the boundary layer in (1.4) (the second relation) is constructed with allowance for the factors ε^β (see [10, 11]) and ε^γ. The meaning of these factors will become clear from the ensuing discussion.

The role of the functions in (1.4) will become evident from a description of the process of their determination. Expansion (1.4) formally satisfies Eq. (1.1) and boundary conditions (1.2). The functions w_i describe the main part of the solution of problem (1.1)-(1.2). Inserting (1.4) (the first relation) into the first equation of system (1.1) and equating first-order terms in ε to zero, we obtain

$$
L_0 w_i(x, y) = \omega^2 M_0 w_i(x, y), \quad L_0 Z_n(x, y) = \omega^2 M_0 Z_n(x, y) \quad (i = 1, 2, 3, \ldots, n). \tag{1.5}
$$

It is evident that the functions w_i and their derivatives are invariant with respect to ε, since the first equation of (1.1) is independent of ε, i.e., its terms are of the same order.

The boundary-layer part of the asymptote consists of two types of boundary functions: Q and R. In the neighborhood of each side Γ_ℓ of the rectangular plate we construct ordinary boundary layers $Q_{\ell i}$ which are described by ordinary differential equations and are boundary-layer functions in one variable. For example, in the neighborhood of Γ_1, $Q_{\ell i}(\xi_1, y)$ is a boundary-layer function in the variable ξ_1, etc.

$$
Q_{li}(\xi_1, y) \to 0 \quad \text{at} \quad \xi_1 \to \infty. \tag{1.6}
$$

The boundary functions $Q_{\ell i}$ change as follows in relation to ε:

$$
\frac{\partial^{|\alpha|}}{\partial n^{\alpha_1} \partial s^{\alpha_2}} Q_{li} \sim \varepsilon^{-\alpha_1} Q_{li}, \quad Q_{li} \sim O(1), \quad |\alpha| = \alpha_1 + \alpha_2.
$$

In accordance with [6], boundary functions $R_{\ell i}$ in two variables determined from elliptic equations are introduced in the neighborhood of points of inflection. Thus, in the neighborhood of a vertex (0, 0), $R_{\ell i}(\xi_1, \eta_1)$ is a boundary-layer function in the variables ξ_1 and η_1:

$$
R_{li}(\xi_1, \eta_1) \to 0 \quad \text{at} \quad \xi_1^2 + \eta_1^2 \to \infty. \tag{1.7}
$$

The angular boundary layer $R_{\ell i}(\xi, \eta)$ changes as follows in relation to ε:

$$
\frac{\partial^{|\alpha|}}{\partial n^{\alpha_1} \partial s^{\alpha_2}} R_{li} \sim \varepsilon^{-|\alpha|} R_{li}, \quad R_{li} \sim O(1).
$$

Inserting (1.4) (the second relation) into the second equation of system (1.1) and equating to zero the terms of the first order with respect to ε for the functions $Q_{\ell i}$, $R_{\ell i}$, and z_n, we obtain the following iterative chain of equations for Q_{1i}:

$$
\begin{gathered}
\partial^2 Q_{1i}/\partial \xi_1^2 - Q_{1i} = g_{1i}(\xi_1, y) \quad (\xi_1 > 0, \ 0 \leqslant y \leqslant b/a), \\
g_{1i}(\xi_1, y) = -\partial^2 Q_{1i-2}/\partial y^2, \ g_{10} = g_{11} = 0 \ (i = 0, 1, 2, \ldots, n).
\end{gathered}
\tag{1.8}
$$

Here, the variable y is a parameter.

For $R_{\ell i}(\xi_1, \eta_1)$

$$
\left(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \eta_1^2} - 1 \right) R_{1i}(\xi_1, \eta_1) = 0 \quad (\xi_1 > 0, \ \eta_1 > 0). \tag{1.9}
$$

The boundary functions $Q_{\ell i}$, $R_{\ell i}$ (ℓ = 2, 3, 4) are determined from similar equations in the neighborhood of the other sides and corner points. For the remainder $z_n(x, y)$

$$(\varepsilon^2\Delta - 1)\, z_n(x, y) = h(x, y), \quad h(x, y) = \sum_{l=1}^{n}\sum_{k=0}^{1} \varepsilon^k \partial^2 Q_{l,n-1+k}/\partial y^2. \tag{1.10}$$

The specific form of the boundary layers is obtained with allowance for the boundary conditions. As an example, we will examine the side Γ_1. Inserting expansion (1.4) into boundary conditions (1.2) and ignoring the mutual effect of the boundary layers (i.e., taking only the functions Q_{1i}, Q_{2i}, Q_{4i}, R_{1i}, R_{4i} into account in the boundary conditions on Γ_1), we find

$$B_k\left(\sum_{i=0}^{n} \varepsilon^i w_i + \varepsilon^{n+1} Z_n\right) + \varepsilon^{\beta}\Bigg[\sum_{i=0}^{n} \varepsilon^i \big(\varepsilon^{-\alpha_1} a_{kl}(\varepsilon)\, D_1^{\alpha} Q_{1i} +$$
$$+ \varepsilon^{-\alpha_2} a_{kl}(\varepsilon)\, D_2^{\alpha}(Q_{2i} + Q_{4i}) + \varepsilon^{\gamma-|\alpha|} a_{kl}(\varepsilon)\, D_3(R_{1i} + R_{4i})\big) + \varepsilon^{n+1} z_n\Bigg]\Bigg|_{\Gamma} = 0. \tag{1.11}$$

Here, $D_1^{\alpha} = \partial^{|\alpha|}/\partial\xi_1^{\alpha_1}\partial y^{\alpha_2}$; $D_2^{\alpha} = \partial^{|\alpha|}/\partial x^{\alpha_1}\partial\eta^{\alpha_2}$; $D_3^{\alpha} = \partial^{|\alpha|}/\partial\xi_1^{\alpha_1}\partial\eta^{\alpha_2}$.

It is necessary to keep in mind that in boundary conditions (1.11) the boundary layers are written in a transformed coordinate system. It should be noted that only the last condition in (1.11) k = m + 1 contains the lowest degree of the small parameter ε as a factor. By virtue of conditions A and B, in the zeroth step of the iteration the functions Q_{1i}, R_{1i}, and R_{4i} should compensate for the error in the last boundary condition of (1.11) k = m + 1

$$\left[B_{m+1} w_0 + \varepsilon^{\beta - q_{m+1}} T^0_{m+1} Q_{10}\right] + \varepsilon^{\beta}\left[\varepsilon^{-\varkappa_2} T^2_{m+1} Q_{20} + \varepsilon^{\gamma-\varkappa_1} T^1_{m+1} R_{10}\right] +$$
$$+ \varepsilon^{\beta}\left[\varepsilon^{-\varkappa_2} T^2_{m+1} Q_{40} + \varepsilon^{\gamma-\varkappa_1} T^1_{m+1} R_{40}\right]\big|_{\Gamma_1} = 0, \tag{1.12}$$

where $\varkappa_1 = \max\limits_{\alpha}(|\alpha| - p_{m+1,\alpha})$; $\varkappa_2 = \max\limits_{\alpha}(\alpha_2 - p_{m+1,\alpha})$. The operators $T^1_{m+1}(T^2_{m+1})$ contain only those derivatives in which the coefficients are of the order $\varepsilon^{-\varkappa_1}(\varepsilon^{-\varkappa_2})$.

In accordance with [6], the function Q_{10} eliminates the error for w_0, while $R_{10}(R_{40})$ eliminates the error introduced by the function $Q_{20}(Q_{40})$ in boundary condition (1.12) in the neighborhood of the point of inflection (0, 0) [the point (0, a)]. Thus, condition (1.12) takes the simpler form

$$\varepsilon^{\beta - q_{m+1}}\left(T^0_{m+1} Q_{10}\right)\big|_{\Gamma_1} = -B_{m+1} w_0\big|_{\Gamma}. \tag{1.13}$$

The main part of the solution w_0 has already been constructed. Similar conditions can be found for the functions $Q_{\ell 0}$ on Γ_{ℓ}.

For the functions R_{10} on Γ_1 we obtain

$$\varepsilon^{\gamma-\varkappa_1} T^1_{m+1} R_{10}\big|_{\Gamma_1} = -\varepsilon^{-\varkappa_2} T^2_{m+1} Q_{20}\big|_{\Gamma_1}. \tag{1.14}$$

We have conditions similar to (1.14) for R_{10} on Γ_2. The boundary conditions for the other angular boundary layers $R_{\ell 0}$ near the other vertices of the rectangle are constructed in a similar manner.

To properly construct the iteration process, quantities of the same order of smallness should remain in the left and right sides of (1.13) and (1.14). We thus write the parameters β and γ in expansion (1.4) in the form

$$\beta = q_{m+1}, \ \gamma = \varkappa_1 - \varkappa_2.$$

We then assemble terms with the same degrees of the small parameter in Eqs. (1.11) (k = 1, 2, 3,...,m). First we collect the terms with ε to the zeroth degree:

$$B_k w_0\big|_{\Gamma_l} = \Phi_k(Q_{10}, Q_{20}, Q_{40}, R_{10}, \ldots)\big|_{\Gamma_l}. \tag{1.15}$$

The right side of conditions (1.15) is a differential operator of the boundary-layer functions determined in the neighborhood of Γ_{ℓ} and the adjacent sides. By virtue of conditions (1.13) on Γ_1 and similar conditions on Γ_{ℓ}, the functions $Q_{\ell 0}$ are found through w_0. It follows from (1.14) and the analogous conditions in the neighborhoods of the other points of inflection that $R_{\ell 0}$ is also determined through w_0. Taking the formulas expressing $Q_{\ell 0}$ and $R_{\ell 0}$ in terms of w_0 and inserting them into the right sides of (1.15), we arrive at boundary conditions for w_0

$$B_k w_0 - \Phi_k^0(w_0)\big|_{\Gamma_l} = 0 \quad (k = 1, 2, \ldots, m). \tag{1.16}$$

Thus, we have problem (1.5), (1.16) to determine w_0. We will refer to this as the truncated problem. Knowing its solution, we can easily reconstruct the boundary layer. Thus, the zeroth approximation of w_0, $Q_{\ell 0}$, and $R_{\ell 0}$ in the initial problem has been found.

Placing the terms with ε to the first degree in (1.11), we obtain boundary conditions to determine the boundary layers $Q_{\ell 1}$ and $R_{\ell 1}$. These conditions are similar to (1.13) and (1.14), except that the right side is dependent on w_0, w_1, $Q_{\ell 0}$, and $R_{\ell 0}$. The problems for the subsequent approximations are formulated in a similar manner.

The remainders satisfy the boundary conditions

$$B_k Z_n + \varepsilon^{\beta} \sum_{|\alpha|=0}^{N_k} a_{k\alpha}(\varepsilon) D^{\alpha} z_n |_{\Gamma} = \sum_{i=0}^{n} \varepsilon^i A_i (Q_{li}, R_{li}) \qquad (k = 1, 2, 3, \ldots, m, m+1) \tag{1.17}$$

(A_i are certain differential operators). In accordance with condition A, problem (1.5), (1.10), (1.17) has a unique solution.

Let us proceed to the study of specific problems.

2. We will examine a system of equations describing the free vibration of a transversely isotropic rectangular plate:

$$\Delta\Delta w(x, y) - \omega^2 k^2 (1 - \vartheta \Delta) w(x, y) = 0; \tag{2.1}$$

$$\varepsilon^2 \Delta v(x, y) - v(x, y) = 0. \tag{2.2}$$

The following dimensionless quantities were introduced in (2.1)-(2.2) [1]: w(x, y) is the deflection of the plate; v(x, y) is the resolvent function; ω is the frequency of vibration; k is a large parameter characterizing the bending stiffness of the plate; ϑ is the corrected thickness of the plate; ε is a small parameter characterizing the shear stiffness of the plate,

$$\omega = \omega^* a \sqrt{\rho/E}, \quad k^2 = 12(1 - \nu^2) a^2/h^2, \quad \vartheta = \left(2\frac{G}{G'} - \nu' \frac{E}{E'}\right) \frac{h^2}{10a^2(1 - \nu)}, \quad \varepsilon^2 = \frac{h^2}{10a^2} \frac{G}{G'}, \tag{2.3}$$

where ω* is the dimensionless frequency of vibration; a, b, and h are the lengths of the sides and the thickness of the plate; ρ is density; ν, G, and E are the Poisson's ratio, shear modulus, and Young's modulus in the plane of isotropy.

In Eqs. (2.3), k, ϑ, and ε include the same powers of h/a. However, they must still be distinguished from one another by virtue of their different physical meanings. Although it would undoubtedly be interesting to find the asymptotic solutions for several small parameters, this problem is not examined here.

The dimensionless moments, forces, and other quantities are determined from the relations

$$M_n = \frac{1}{Eh} M_n^*, \quad M_{ns} = \frac{1}{Eh} M_{ns}^*, \quad N_n = \frac{a}{Eh} N_n^*,$$

$$v = \frac{1}{Eh} v^*, \quad \alpha_n = \frac{ah^2}{12E} \alpha_n^*, \quad \alpha_s = \frac{ah^2}{12E} \alpha_s^*.$$

Here, M_n^* and M_{ns}^* are the bending and turning moments; N_n^* is the shearing force; v^* is the resolvent function; α_n^*, α_s^* are the shears along a normal to the boundary and along the boundary respectively. The quantities M_n, M_{ns}, N_n, α_n, α_s can be represented in terms of the sought functions w and v:

$$\begin{aligned} M_n|_{\Gamma_l} &= -\left(\frac{1}{k^2} M_{0n} + \varepsilon^2 B_1 w + (-1)^l 2\varepsilon^2 \frac{\partial^2 v}{\partial n \partial s}\right)_{\Gamma_l}, \\ M_{ns}|_{\Gamma_l} &= -\left(\frac{1}{k^2} M_{0ns} + \varepsilon^2 B_2 w + (-1)^l \left(v - 2\varepsilon^2 \frac{\partial^2 v}{\partial n^2}\right)\right)_{\Gamma_l}, \\ N_s|_{\Gamma_l} = \alpha_n|_{\Gamma_l} &= \left(-\frac{1}{k^2} N_{0s} + (-1)^l \frac{\partial v}{\partial n}\right)_{\Gamma_l}, \\ N_n|_{\Gamma_l} = \alpha_s|_{\Gamma_l} &= \left(-\frac{1}{k^2} N_{0n} - (-1)^l \frac{\partial v}{\partial s}\right)_{\Gamma_l}, \end{aligned} \tag{2.4}$$

where

$$\Delta^* = \frac{\partial}{\partial n^2} + \nu \frac{\partial}{\partial s^2}; \quad B_1 = \frac{2}{k^2(1 - \nu)} \Delta\Delta^* - 2\gamma \frac{1 + \nu}{1 - \nu} \omega^2 \Delta^* - \frac{\nu' E G' \omega^2}{E' G (1 - \nu)};$$

TABLE 1

No.	Boundary conditions of system (2.1)-(2.2)	Boundary conditions of the truncated problem (zeroth approximation)	β, γ
1	$w=0,\quad M_n=0,\quad M_{ns}=0$	$w_0=0,\quad M_{0n}+k^2F(w_0)=0$ *	0,0
2	$w=0,\ M_n=0,\quad \alpha_n=0$	$w_0=0,\quad M_{0n}=0$	1,1
3	$w=0,\quad M_{ns}=0,\quad -\frac{\partial w}{\partial n}+\varkappa\alpha_s=0$	$w_0=0,\quad -\frac{\partial w_0}{\partial n}+$ $+\varkappa\frac{1}{k^2}\left[N_{0n}+\frac{\partial M_{0ns}}{\partial s}\right]=0$	0,0
4	$w=0,\quad \alpha_n=0,\quad -\frac{\partial w}{\partial n}+\varkappa\alpha_s=0$	$w_0=0,\quad -\frac{\partial w_0}{\partial n}+\varkappa\frac{1}{k^2}N_{0n}=0$	1,1
5	$N_n=0,\quad M_n=0,\quad M_{ns}=0$	$N_{0n}+\frac{\partial M_{0ns}}{\partial s}=0,\quad M_{0n}+$ $+k^2F(w_0)=0$ *	0,0
6	$N_n=0,\quad M_n=0,\quad \alpha_n=0$	$N_{0n}=0,\quad M_{0n}=0$	1,1
7	$N_n=0,\quad M_{ns}=0,\quad -\frac{\partial w}{\partial n}+\varkappa\alpha_s=0$	$N_{0n}+\frac{\partial M_{0ns}}{\partial s}=0,\quad \frac{\partial w_0}{\partial n}=0$	0,0
8	$N_n=0,\quad \alpha_n=0,\quad -\frac{\partial w}{\partial n}+\varkappa\alpha_s=0$	$N_{0n}=0,\quad \frac{\partial w_0}{\partial n}=0$	1,1

* $F(w)=(-1)^l\, 2\,[M_{0ns}|_{\Gamma_l\cap\Gamma_{l+1}} f_{l1}(s)+M_{0ns}|_{\Gamma_l\cap\Gamma_{l-1}} f_{l2}(s)]$.

$$B_2=\frac{2}{k^2}\frac{\partial^2}{\partial n\,\partial s}\Delta+2\gamma(1+\nu)\omega^2\frac{\partial^2}{\partial n\,\partial s},$$

while M_{0n}, M_{0ns}, and N_{0n} are the bending and turning moments and the shearing force corresponding to the classical theory:

$$M_{0n}=\Delta^* w,\quad M_{0s}=(1-\nu)\frac{\partial^2 w}{\partial n\,\partial s},$$

$$N_{0n}=\frac{\partial}{\partial n}\Delta w+k^2\omega^2\gamma(1+\nu)\frac{\partial w}{\partial n},\quad N_{0s}=\frac{\partial}{\partial s}\Delta w+k^2\omega^2\gamma(1+\nu)\frac{\partial w}{\partial s}.$$

The boundary conditions for system (2.1)-(2.2) are shown in Table 1. First let us examine boundary condition 1, corresponding to hinged support of the edge. Using (2.4), we obtain the following on Γ_ℓ

$$w|_{\Gamma_l}=0; \tag{2.5a}$$

$$\frac{1}{k^2}M_{0n}+\varepsilon^2 B_1 w+(-1)^l 2\varepsilon^2\frac{\partial^2 v}{\partial n\,\partial s}\Big|_{\Gamma_l}=0; \tag{2.5b}$$

$$\frac{1}{k^2}M_{0ns}+\varepsilon^2 B_2 w+(-1)^l\left(v-2\varepsilon^2\frac{\partial^2 v}{\partial n^2}\right)\Big|_{\Gamma_l}=0. \tag{2.5c}$$

It is evident that the ε-th-order effect of conditions (2.5b)-(2.5c) is equal to $q_2=-1$, $q_3=0$. This means that the boundary layer should eliminate the error in the last condition.

We insert expansion (1.4) into Eqs. (2.1)-(2.2) and boundary conditions (2.5). In accordance with Part 1, we have the following problem for the zeroth approximation: for Q_{10} — Eq. (1.8) ($i=0$), condition at infinity (1.6), and the boundary condition on Γ_1

$$\varepsilon^2\left(2\frac{\partial^2 Q_{10}}{\partial\xi_1^2}-Q_{10}\right)\Big|_{\Gamma_1}=(-1)^l\frac{1}{k^2}M_{0xy}^{(0)}|_{\Gamma_1}. \tag{2.6}$$

Here and below, $M_{0n}^{(i)}, M_{0ns}^{(i)}, N_{0n}^{(i)}, N_{0s}^{(i)}$ corresponds to the i-th term of expansion (1.4).

As an example, we will examine the function R_{10}. The latter is determined from Eq. (1.9), condition at infinity (1.7), and the boundary conditions

$$\varepsilon^{\gamma}\left(2\frac{\partial R_{10}}{\partial \xi_1^2} - R_{10}\right)_{\Gamma_1} = Q_{20}|_{\Gamma_1}, \quad \varepsilon^{\gamma}\left(2\frac{\partial}{\partial}\frac{R_{10}}{\eta_1^2} - R_{10}\right)_{\Gamma_2} = Q_{10}|_{\Gamma_2}. \tag{2.7}$$

In (2.6)-(2.7), $\beta = 0$, $\gamma = 0$. The function w_0 is found from Eq. (2.1) and the boundary conditions. Thus, on Γ_ℓ

$$w_0|_{\Gamma_l} = 0, \ \frac{1}{k^2}M_{0n}^{(0)} - 2\left(\frac{\partial^2 R_{10}}{\partial \xi_1 \partial \eta_1} + \frac{\partial^2 R_{40}}{\partial \xi_1 \partial \eta_2}\right)_{\Gamma_l} = 0. \tag{2.8}$$

We now need to establish the specific form of the boundary layers Q_{10} and Q_{20}:

$$\begin{gathered} Q_{10}(\xi_1, y) = p_{10}(y)\exp(-\xi_1), \quad p_{10}(y) = -\frac{1}{k^2}M_{0xy}^{(0)}|_{\Gamma_1}, \\ Q_{20}(x, \eta_1) = p_{20}(x)\exp(-\eta_1), \quad p_{10}(x) = -\frac{1}{k^2}M_{0xy}^{(0)}|_{\Gamma_2}. \end{gathered} \tag{2.9}$$

Inserting (2.9) into (2.7), we obtain the boundary conditions for R_{10}

$$\begin{gathered} \left(2\frac{\partial^2 R_{10}}{\partial \xi_1^2} - R_{10}\right)_{\Gamma_1} = p_{20}(0)\exp(-\eta_1), \\ \left(2\frac{\partial^2 R_{10}}{\partial \eta_1^2} - R_{10}\right)_{\Gamma_2} = p_{10}(0)\exp(-\xi_1). \end{gathered} \tag{2.10}$$

It follows from (2.9) that there is a discontinuity at the corner point in boundary conditions (2.10), i.e.,

$$p_{20}(0) - p_{10}(0) = \frac{2}{k^2}M_{0xy}^{(0)}|_{\Gamma_1 \cap \Gamma_2}.$$

The Green's function of problem (1.9), (1.7), (2.10) is constructed by the transform method and has the form

$$G(\xi, \eta, \tau, t) = (1/2\pi)[K_0(r_1) + K_0(r_2) - K_0(r_3) - K_0(r_4)], \tag{2.11}$$

where $K_0(r)$ is a cylinder function of the imaginary argument:

$$\begin{gathered} r_1 = [(\xi - \tau)^2 + (\eta - t)^2]^{1/2}, \ r_2 = [(\xi + \tau)^2 + (\eta + t)^2]^{1/2}, \\ r_3 = [(\xi - \tau)^2 + (\eta + t)^2]^{1/2}, \ r_4 = [(\xi + \tau)^2 + (\eta - t)^2]^{1/2}. \end{gathered}$$

Then the solution of problem (1.9), (1.7), (2.10) is represented as

$$\begin{gathered} R_{10}(\xi, \eta) = p_{20}(0)I_1(\xi, \eta) + p_{10}(0)I_2(\xi, \eta), \\ I_1 = \int_{\Gamma_1} \frac{\partial G(\xi, \eta, \tau, t)}{\partial \tau}\bigg|_{\tau=0} \exp(-t)\,dt, \ \ I_2 = \int_{\Gamma_2} \frac{\partial G(\xi, \eta, \tau, t)}{\partial t}\bigg|_{t=0} \exp(-\tau)\,d\tau. \end{gathered} \tag{2.12}$$

The functions $R_{\ell 0}$ are constructed similarly in the neighborhood of the other corner points. Inserting R_{10} and R_{40} into boundary conditions (2.8) and taking into account Eqs. (2.9) and (2.12), we obtain boundary conditions for w_0 on Γ_1. Finally, the boundary conditions for the zeroth approximation can be written in the form

$$\frac{1}{k^2}M_{0n}^{(0)}|_{\Gamma_l} = (-1)^l 2\left[M_{0n}^{(0)}|_{\Gamma_l \cap \Gamma_{l+1}} f_{l1}(s) + M_{0n}^{(0)}|_{\Gamma_l \cap \Gamma_{l-1}} f_{l2}(s)\right]; \tag{2.13a}$$

$$w_0|_{\Gamma_l} = 0, \tag{2.13b}$$

$$f_{lk} = \sum_{p=1}^{2} \frac{\partial^2 I_p(\xi_l, \eta_k)}{\partial \xi_l \partial \eta_k} \ \ (l=1, 3), \quad f_{lk} = \sum_{p=1}^{2} \frac{\partial^2 I_p(\xi_k, \eta_l)}{\partial \xi_k \partial \eta_l} \ \ (l = 2, 4).$$

In Eq. (2.13), we assumed that $\Gamma_0 = \Gamma_4$ and $\Gamma_5 = \Gamma_1$. As was proven in [6], the function $f_{\ell k}(s)$ has exponential values [6, 7]

$$|f_{11}(\eta_1)| \leqslant C\exp(-\delta\eta_1), \ |f_{12}(\eta_2)| \leqslant C\exp(-\delta\eta_2) \tag{2.14}$$

($0 < \delta \leqslant 1$, C and δ are arbitrary constants).

For w_0, we obtain a generalized eigenfunction problem and the numbers (2.1), (2.13). We will refer to this as the truncated problem (zeroth approximation) of the initial problem (2.1)-(2.2) with the boundary conditions shown in Table 1. It follows from (2.14) that the right sides of (2.13) differ appreciably from zero only near the points of inflection. In the simplified analysis of the truncated problem, the effect of the correction factors $f_{k\ell}(s)$ in the boundary conditions can be ignored. Let us assume that the solution of the truncated problem has been found. We then establish the edge effects in expansion (1.4), thus completing the construction of the solution (zeroth approximation).

Let us now proceed to the construction of the subsequent approximations w_i and v_i of problem (2.1)-(2.2) with the boundary conditions from Table 1 ($i = 1, 2, 3,\ldots,n$). Inserting expansion (1.4) into the equation and the boundary conditions of the initial problem and equating terms of the same order of smallness to zero, we obtain the following problem.

For Q_{i1}, we have Eq. (1.8), boundary conditions (1.6), and

$$2\frac{\partial^2 Q_{1i}}{\partial \xi_1^2} - Q_{1i}|_{\Gamma_1} = -\frac{1}{k^2} M_{0xy}^{(i)} - B_2 w_{i-2}|_{\Gamma_1}. \tag{2.15}$$

The angular boundary layer R_{1i} is determined from Eq. (1.9), boundary conditions (1.7), and

$$\begin{aligned} 2\frac{\partial^2 R_{1i}}{\partial \xi_1^2} - R_{1i}|_{\Gamma_1} &= Q_{2i} - 2\left.\frac{\partial^2 Q_{2,i-2}}{\partial x^2}\right|_{\Gamma_1}, \\ 2\frac{\partial^2 R_{1i}}{\partial \eta_1^2} - R_{1i}|_{\Gamma_2} &= Q_{1i} - 2\left.\frac{\partial^2 Q_{2,i-2}}{\partial y^2}\right|_{\Gamma_2}. \end{aligned} \tag{2.16}$$

The function w_i is determined from Eq. (2.1) and the condition on the boundary. Thus, on Γ_1

$$\frac{1}{k^2} M_{0n}^{(i)} - 2\left.\left(\frac{\partial^2 R_{1i}}{\partial \xi_1 \partial \eta_1} + \frac{\partial^2 R_{1i}}{\partial \xi_1 \partial \eta_2}\right)\right|_{\Gamma_1} = -B_1 w_{i-2} + 2[Q_{1i-1} + Q_{2i-1} + Q_{4i-1}], \quad w_i|_{\Gamma_1} = 0. \tag{2.17}$$

Here and below, the expressions with negative indices are identically equal to zero.

We seek the solution of problem (1.8), (1.6), (2.16) in the form

$$Q_{10}(\xi_1, y) = [p_{10}(y) + \beta_{1i}(\xi_1, y)]\exp(-\xi_1), \tag{2.18}$$

where the first term is the solution of homogeneous equation (1.8) with inhomogeneous boundary conditions (2.16). The second term is the solution of inhomogeneous equation (1.8) with homogeneous boundary conditions (2.16). Then

$$\begin{aligned} p_{1i}(y) &= -\left(\frac{1}{k^2} M_{0n}^{(i)} - B_2 w_{i-2}\right)_{\Gamma_1}, \\ \beta_{1i}(\xi_1, y) &= -\int_0^{\xi_1} \mathrm{sh}(\tau)\, g_{1i}(\tau, y)\, d\tau - \int_{\xi_1}^{\infty} \mathrm{sh}(\xi_1)\exp(\xi_1 - \tau)\, g_{1i}(\tau, y)\, d\tau. \end{aligned} \tag{2.19}$$

It is not hard to show [see (1.8)] that the discontinuity $g_{1i}(\xi_1, y) = \varphi_i(\xi_1, y)\exp(-\xi_1)$ ($\varphi_i(\xi_1, y)$ is a polynomial in ξ_1 with coefficients dependent on y).

Thus, $\beta_{1i}(\xi_1, y)$ is a bounded function in ξ_1, and the following exponential estimate holds for Q_{1i}

$$|Q_{1i}(\xi_1, y)| \leqslant C(y)\exp(-\delta\xi_1).$$

With allowance for (2.16), (2.18), the angular boundary layer is determined from the following conditions on the boundary. Thus, for R_{1i}

$$\begin{aligned} 2\frac{\partial^2 R_{1i}}{\partial \xi_1^2} - R_{1i}|_{\Gamma_1} &= [p_{2i}(0) + \varkappa_{2i}(\eta_1)]\exp(-\eta_1), \\ 2\frac{\partial^2 R_{1i}}{\partial \eta_1^2} - R_{1i}|_{\Gamma_2} &= [p_{1i}(0) + \varkappa_{1i}(\xi_1)]\exp(-\xi_1), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} \varkappa_{1i}(\xi_1) &= \beta_{1i}(\xi_1, 0) + [\beta_{1i-2}(\xi_1, 0) + p_{1i-2}(0)]_{yy}; \\ \varkappa_{2i}(\eta_1) &= \beta_{2i}(0, \eta_1) + [\beta_{1i-2}(0, \eta_1) + p_{2i-2}(0)]_{xx}. \end{aligned}$$

Here, as for the zeroth approximation, the boundary conditions are discontinuous at the corner point. The magnitude of the first-order discontinuity is determined from (2.15) and the analogous condition for $Q_{\ell i}$. Restrictions on the function $p_{\ell i}(s)$ and, thus, on w_i follow from (2.20): the functions $p_{\ell i}(s)$ are bounded derivatives $p_{\ell i,ss}(s)|_{s=0\pm 0}$ at i = 0, 1, 2,..., n − 2.

The solution of problem (1.9), (1.7), (2.20) is found from formulas similar to Eqs. (2.12):

$$R_{1i}(\xi, \eta) = p_{2i}(0)I_1(\xi, \eta) + p_{1i}(0)I_2(\xi, \eta) + T_{2i}(\xi, \eta) + T_{1i}(\xi, \eta),$$
$$T_{1i} = \int\limits_{\Gamma_1} \frac{\partial G(\xi, \eta, \tau, t)}{\partial \tau}\bigg|_{\tau=0} \varkappa_{2i}(t)\exp(-t)\,dt,$$
$$T_{2i} = \int\limits_{\Gamma_2} \frac{\partial G(\xi, \eta, \tau, t)}{\partial t}\bigg|_{t=0} \varkappa_{1i}(\tau)\exp(-\tau)\,d\tau.$$

Expressions for $I_k(\xi, \eta)$ are contained in (2.12).

As a result, we obtain the boundary conditions for the truncated problem (the i-th approximation):

$$w_i|_{\Gamma_l} = 0; \tag{2.21a}$$

$$\frac{1}{k^2} M_{0n}^{(i)}|_{\Gamma_l} + (-1)^l 2\left[M_{0ns}^{(i)}|_{\Gamma_l\cap\Gamma_{l+1}} f_{l1}(s) + M_{0ns}^{(i)}|_{\Gamma_l\cap\Gamma_{l-1}} f_{l2}(s)\right] =$$
$$= B_2 w_{i-2} + 2Q_{1i-1} + g_{l1i}(s) + g_{l2i}(s),$$
$$g_{lki}(s) = \left(\frac{\partial^2 T_{li}(\xi_l, \eta_k)}{\partial \xi_l\, \partial \eta_k}\right)_{\Gamma_l} \quad (l = 1, 3), \tag{2.21b}$$
$$g_{lki}(s) = \left(\frac{\partial^2 T_{li}(\xi_k, \eta_l)}{\partial \xi_k\, \partial \eta_l}\right)_{\Gamma_l} \quad (l = 2,4).$$

The functions $g_{\ell ki}(s)$ have exponential estimates of the type (2.14) and differ significantly from zero only in the neighborhood of the corner points.

Knowing the solution of the truncated problem (i-th approximation), we reconstruct the boundary effects. The truncated problem (i-th approximation) is found on a spectrum and should thus satisfy certain solvability conditions (which are not presented here).

As can be seen from Table 1, the boundary conditions are divided into two groups: 1, 3, 5, 7, and 2, 4, 6, 8. The solution of the initial problem with boundary conditions of the first group is constructed in a manner analogous to problem (2.1)-(2.2), with boundary conditions from Table 1. It is easily seen that formulas (2.6)-(2.7), (2.9)-(2.14) are valid for the boundary-layer part of the asymptote for the zeroth approximation, while formulas (2.15)-(2.16), (2.18)-(2.21) are valid for subsequent approximations. The boundary conditions of the truncated problem are found by the elimination method (see Part 1). We will take a closer look at the case of edge 5, free of constraints. The first boundary condition of the truncated problem will coincide with (2.13a) (i = 0) and (2.21b) (i = 1, 2,...,n), while the second boundary condition on Γ_1 will have the form

$$N_{0n}^{(i)}|_{\Gamma_l} = -k^2 \frac{\partial}{\partial y}\left[Q_{1i} + Q_{2i} + Q_{4i} + R_{1i} + R_{4i}\right]_{\Gamma_l}.$$

Inserting $Q_{\ell i}$ and $R_{\ell i}$ into this expression, we find for the zeroth approximation w_0 that

$$N_{0n}^{(i)}|_{\Gamma_1} = -k^2\left[\frac{\partial}{\partial y} p_{10}(y) - \frac{1}{\varepsilon} p_{20}(0)\exp(-\eta_1) + \frac{1}{\varepsilon} p_{20}(0)\frac{\partial I_1(0, \eta_1)}{\partial \eta_1} + \right.$$
$$\left. + \frac{1}{\varepsilon} p_{10}(0)\frac{\partial I_2(0, \eta_1)}{\partial \eta_1} - \frac{1}{\varepsilon} p_{40}(0)\exp(-\eta_2) + \frac{1}{\varepsilon} p_{40}(0)\frac{\partial I_1(0, \eta_2)}{\partial \eta_2} + \frac{1}{\varepsilon} p_{10}(0)\frac{\partial I_2(0, \eta_2)}{\partial \eta_2}\right].$$

By direct differentiation, we find from (2.11) that

$$\frac{\partial}{\partial \eta}\left[\frac{\partial G(\xi, \eta, \tau, t)}{\partial \tau}\bigg|_{\tau=0}\right]_{\xi=0} = \delta(\eta - t), \quad \frac{\partial}{\partial \eta}\left[\frac{\partial G(\xi, \eta, \tau, t)}{\partial t}\bigg|_{t=0}\right]_{\xi=0} = 0,$$

where $\delta(\eta - t)$ is the Dirac delta function. It then follows from (2.12) that

$$\frac{\partial I_1(0,\eta)}{\partial\eta}=\exp(-\eta),\quad \frac{\partial I_2(0,\eta)}{\partial\eta}=0. \tag{2.22}$$

With allowance for Eqs. (2.22), (2.9) and the corresponding expressions for the other sides of the plate, we finally write the boundary condition for the zeroth approximation:

$$N_{0n}^{(i)}|_{\Gamma_1}=-k^2(-1)^l\frac{\partial}{\partial s}\,p_{l0}(s)|_{\Gamma_l}=-\frac{\partial}{\partial s}M_{0ns}^{(0)}|_{\Gamma_l}. \tag{2.23}$$

Boundary condition (2.23) corresponds to a generalized Kirchhoff force.

We similarly obtain the boundary conditions of the truncated problem (i-th approximation): the first coincides with (2.12b), while the second has the form

$$N_{0n}^{(i)}+\frac{\partial}{\partial s}M_{0ns}^{(i)}|_{\Gamma_1}=-\frac{\partial}{\partial y}\beta_{1i}(y)-\frac{1}{\varepsilon}\sum_{k=1}^{2}\left(\beta_{p_k i}(0,\eta)-\varkappa_{p_k i}(\eta)\right)\ \exp(-\eta)\qquad (p_1=2,\ p_2=4).$$

Thus, we obtain the Kirchhoff transform in the neighborhood of a corner point, i.e., for a piecewise-smooth contour.

Let us proceed to boundary conditions 2, 4, 6, 8 (see Table 1). As an example, we will examine conditions 2. The latter correspond to hinged support of an edge with a rigid diaphragm preventing shear of the sandwich:

$$w_l|_{\Gamma_l}=0; \tag{2.24a}$$

$$\frac{1}{k^2}M_{0n}+\varepsilon^2B_1w+(-1)^l\,2\varepsilon^2\frac{\partial^2 v}{\partial n\,\partial s}\Big|_{\Gamma_l}=0; \tag{2.24b}$$

$$\frac{1}{k^2}N_{0s}-(-1)^l\frac{\partial v}{\partial n}\Big|_{\Gamma_l}=0. \tag{2.24c}$$

Here, the ε-th-order effect of boundary conditions (2.24b)-(2.24c) is equal to $q_2=-1$, $q_3=1$. In accordance with Part 1, the boundary layer is determined from (2.24c). Thus, for Q_{1i} we have the boundary conditions

$$\varepsilon^{\beta-1}\frac{\partial Q_{1i}}{\partial\xi_1}\Big|_{\Gamma_1}=(-1)^l\frac{1}{k^2}N_{0s}^{(i)}|_{\Gamma_1}.$$

The angular boundary layer R_{1i} is determined from Eq. (1.9) and the boundary conditions

$$\varepsilon^{\gamma-1}\frac{\partial R_{1i}}{\partial\xi_1}\Big|_{\Gamma_1}=-\frac{\partial Q_{2i}}{\partial x}\Big|_{\Gamma_1},\quad \varepsilon^{\gamma-1}\frac{\partial R_{1i}}{\partial\eta_1}\Big|_{\Gamma_2}=-\frac{\partial Q_{1i}}{\partial y}\Big|_{\Gamma_2},$$

where $\beta=1$, $\gamma=1$.

The ordinary boundary layer contains the multiplier ε, while the angular boundary layer contains ε^2. Then we find the functions w_i from Eq. (2.1) and the boundary conditions

$$w_i|_{\Gamma_l}=0,\quad M_{0n}^{(i)}=k^2\left\{B_1w_{i-2}-(-1)^l\left[\frac{\partial^2Q_{l,i-2}}{\partial(n/\varepsilon)\,\partial s}+\frac{\partial^2R_{l,i-2}}{\partial\xi\,\partial\eta}\right]\right\}_{\Gamma_l}. \tag{2.25}$$

The right part for the zeroth and first approximations in (2.25) is identically equal to zero.

Formulas of the type (2.18) are valid for Q_{1i}. We have the following relations for these formulas

$$p_{1i}(y)=-\frac{1}{k^2}N_{0s}^{(i)}|_{\Gamma_l},$$

$$\beta_{1i}(\xi,y)=-\int_0^{\xi_1}\mathrm{ch}(\tau)\,g_{1i}(\tau,y)\,d\tau-\int_{\xi_1}^{\infty}\mathrm{ch}(\xi_1)\,g_{1i}(\tau,y)\exp(\xi_1-\tau)\,d\tau.$$

The following formula is valid for angular boundary layer R_{1i}

$$R_{1i}(\xi,\eta)=-\int_{\Gamma_1}G_1(\xi,\eta,\tau,t)|_{\tau=0}\,[p_{2i,\tau}(0)+\beta_{2i,\tau}(0,t)]\exp(-t)\,dt-$$

$$-\int_{\Gamma_2}G_1(\xi,\eta,\tau,t)|_{t=0}\,[p_{1i,t}(0)+\beta_{1i,t}(\tau,0)]\exp(-\tau)\,d\tau$$

$$\left(G_1(\xi,\eta,\tau,t)=\frac{1}{2\pi}\,[K_0(r_1)+K_0(r_2)+K_0(r_3)+K_0(r_4)]\right).$$

As regards boundary conditions 4, 6, and 8 from Table 1, the procedure for obtaining the boundary conditions of the truncated problem is similar to the procedure used above. Table 1 shows the boundary conditions for the zeroth approximation – which, except for conditions 1 and 5, correspond to the boundary conditions in the classical formulation. It should be pointed out that boundary conditions 3 and 4 include the following additional terms

$$\varkappa \frac{1}{k^2}\left(N_{0n}+\frac{\partial M_{0ns}}{\partial s}\right), \quad \varkappa \frac{1}{k^2}(N_{0n}), \quad \varkappa = \frac{6E}{h^2G'}\left(\frac{h^2}{4}-\frac{z_0^2}{3}\right),$$

these terms being small, since $k^2 \gg 1$. They describe the effect of shear of the entire sandwich through its thickness. It should also be noted that the Kirchhoff transform does not always exist for a free edge. For example, while the former does exist for edge 5 – which is free of constraints – it does not for the free edge with the diaphragm preventing shear of the sandwich. In the second case, it is necessary to assign the shearing force in the classical formulation in place of the generalized Kirchhoff force.

Boundary conditions 1 and 5 contain additional terms of the form

$$M_{0ns}|_{\Gamma_l \cap \Gamma_{l+1}} f_{lk}(s), \quad |f_{lk}(s)| \leqslant C\exp(-\delta s).$$

Their appearance is related only to the presence of the points of inflection, because they would not be present in the boundary conditions of the truncated problem if the boundary were smooth [3-5, 10, 11]. This result can be explained as follows from a mechanical viewpoint. The turning moment $M_{n,s}$ on the side Γ_ℓ in the neighborhood of the corner points is experienced as a bending moment on the adjacent sides $\Gamma_{\ell+1}$ and $\Gamma_{\ell-1}$. Also, the function $f_{\ell k}(s)$ characterizes the depth of penetration of this additional bending moment in the neighborhood of the corner points on the adjacent sides $\Gamma_{\ell+1}$ and $\Gamma_{\ell-1}$.

3. Let us now examine the problem of the free vibration of a three-layer plate [2]

$$(1-\vartheta\mu^2\Delta)\Delta\Delta X(x, y) - \omega^2k^2(1-\mu^2\Delta)X(x, y) = 0; \tag{3.1}$$

$$\varepsilon^2\Delta v(x, y) - v(x, y) = 0 \tag{3.2}$$

(X is the resolvent function; v is a function characterizing the shear of the three-layer sandwich). Here, we introduce the notation

$$\omega = \omega^* a\sqrt{\rho/E}, \quad k^2 = 12(1-\nu^2)a^2/h^2\eta_*,$$

$$\varepsilon^2 = \frac{1-\nu}{2}\frac{h^2}{\beta_* a^2}, \quad \mu^2 = \frac{h^2}{\beta_* a^2},$$

The small parameters ε and μ are of the same order but have different meanings in a mechanical context: the field of the angles of rotation consists of a potential part (corresponding to μ) and a curl part (corresponding to ε). Thus, these quantities must be distinguished from each other.

The following parameters enter into Eqs. (3.3): ω^* is the dimensional frequency of vibration; a, b, and h are the lengths of the sides and the corrected thickness of the plate; E and ν are the corrected Young's modulus and Poisson's ratio [2]. The small parameter ϑ characterizes the natural bending stiffness of the load-bearing layers, while the large parameter k characterizes the bending stiffness of the entire three-layer sandwich. Also, η_* is the mutual location of the layers, β_* is the capacity of the plate to resist a transverse load, and γ_* is a parameter equal to the ratio of the transverse force experienced by the filler to the total transverse force.

Although the ratio h/a is of the same degree in each of the parameters k, ε, and μ, each case is different because the ratio has different physical meanings. We will examine the asymptote only for ε.

We introduce the dimensionless moments, generalized moments, forces, and other quantities:

$$M_n = \frac{1}{Eh}M_n^*, \quad M_{ns} = \frac{1}{Eh}M_{ns}^*, \quad N_n = \frac{a}{Eh}N_n^*,$$

$$H_n = \frac{1}{Eh}H_n^*, \quad H_{ns} = \frac{1}{Eh}H_{ns}^*, \quad \alpha_n = \alpha_n^*, \quad \alpha_s = \alpha_s^*,$$

where M_n and M_{ns} are the bending and turning moments; H_n and H_{ns} are the generalized bending and turning moments; N_n is the shearing force; α_n and α_s are the normal and tangent angles of shear. These quantities are expressed as follows:

TABLE 2

No.	Boundary conditions of system (3.1)-(3.2)	Boundary conditions of Eq. (3.1) (zeroth approximation)	β, γ
1	$w=0,\quad M_n=0,\quad H_n=0,\quad H_{ns}=0$	$w_0=0,\quad H_{0n}+F(w_0)=0^*,$ $M_{0n}+(1-\vartheta)F(w_0)=0$	2,0
2	$w=0,\quad M_n=0,\quad H_n=0,\quad \alpha_n=0$	$w_0=0,\quad M_{0n}=0,\quad H_{0n}=0$	3,1
3	$w=0,\quad \frac{\partial w}{\partial n}=0,\quad \alpha_s=0,\quad H_{ns}=0$	$w_0=0,\quad \frac{\partial w_0}{\partial n}=0,\quad \frac{\partial}{\partial n}\Delta X_0+$ $+\frac{\partial}{\partial s}H_{0ns}=0$	2,0
4	$w=0,\quad \frac{\partial w}{\partial n}=0,\quad \alpha_n=0,\quad \alpha_s=0$	$w_0=0,\quad \frac{\partial w_0}{\partial n}=0,\quad \frac{\partial}{\partial n}\Delta X_0=0$	3,1
5	$N_n=0,\quad M_n=0,\quad H_n=0,\quad H_{ns}=0$	$N_{0n}+(1-\vartheta)\frac{\partial}{\partial s}H_{0ns}=0,$ $M_{0n}+(1-\vartheta)F(w_0)=0^*,$ $H_{0n}+F(w_0)=0^{**}$	2,0
6	$N_n=0,\quad M_n=0,\quad H_n=0,\quad \alpha_n=0$	$N_{0n}=0,\quad M_{0n}=0,\quad H_{0n}=0$	3,1
7	$\frac{\partial w}{\partial n}=0,\quad \alpha_s=0,\quad N_n=0,\quad H_{ns}=0$	$\frac{\partial w_0}{\partial n}=0,\quad N_{0n}+(1-\vartheta)\frac{\partial}{\partial s}H_{0ns}=0,$ $\frac{\partial}{\partial n}\Delta X_0+\frac{\partial}{\partial s}H_{0ns}=0^{**}$	2,0
8	$\frac{\partial w}{\partial n}=0,\quad \alpha_s=0,\quad N_n=0,\quad \alpha_n=0$	$\frac{\partial w_0}{\partial n}=0,\quad N_{0n}=0,\quad \frac{\partial}{\partial n}\Delta X_0=0$	3,1

* $F(w_0)=(-1)^l(1-\vartheta)2\left[H_{0ns}\big|_{\Gamma_l\cap\Gamma_{l+1}}f_{l1}(s)+H_{0ns}\big|_{\Gamma_l\cap\Gamma_{l-1}}f_{l2}(s)\right].$

$$M_n\big|_{\Gamma_l}=-\frac{1}{k^2}\left[M_{0n}+(-1)^l(1-\nu)(1-\vartheta)\frac{\partial^2 v}{\partial n\,\partial s}\right]_{\Gamma_l},$$

$$M_{ns}\big|_{\Gamma_l}=-\frac{1}{k^2}\left[M_{0ns}-(-1)^l\frac{(1-\nu)(1-\vartheta)}{2}\left(2\frac{\partial^2 v}{\partial n^2}-\frac{v}{\varepsilon^2}\right)\right]_{\Gamma_l},$$

$$H_n\big|_{\Gamma_l}=-\frac{\gamma_*}{k^2}\left[H_{0n}+(-1)^l(1-\nu)\frac{\partial^2 v}{\partial n\,\partial s}\right]_{\Gamma_l},$$

$$H_{ns}\big|_{\Gamma_l}=-\frac{\gamma_*}{k^2}\left[H_{0ns}-(-1)^l\frac{(1-\nu)}{2}\left(2\frac{\partial^2 v}{\partial n^2}-\frac{v}{\varepsilon^2}\right)\right]_{\Gamma},$$

$$N_n\big|_{\Gamma_l}=-\frac{\gamma_*}{k^2}\left[N_{0n}+(-1)^l\frac{(1-\nu)(1-\vartheta)}{2}\frac{1}{\varepsilon^2}\frac{\partial v}{\partial s}\right]_{\Gamma_l},$$

$$\alpha_n\big|_{\Gamma_l}=-\frac{1-\vartheta}{\gamma_*}\left[\frac{2}{1-\nu}\frac{\partial}{\partial n}\Delta X-(-1)^l\frac{1}{\varepsilon^2}\frac{\partial v}{\partial n}\right]_{\Gamma_l},$$

$$\alpha_s\big|_{\Gamma_l}=-\frac{1-\vartheta}{\gamma_*}\left[\frac{2}{1-\nu}\frac{\partial}{\partial n}\Delta X-(-1)^l\frac{1}{\varepsilon^2}\frac{\partial v}{\partial s}\right]_{\Gamma_l},$$

$$w=(1-\mu^2\Delta)X,\quad M_{0n}=\Delta^* w,\quad M_{0ns}=(1-\nu)\frac{\partial^2 w}{\partial n\,\partial s},$$

$$H_{0n}=\Delta^* X,\quad H_{0ns}=(1-\nu)\frac{\partial^2 X}{\partial n\,\partial s},\quad N_{0n}=\frac{\partial}{\partial n}\Delta w.$$

The boundary conditions for system (3.1)-(3.2) are shown in Table 2. The fact that the boundary conditions in v for the three-layer and transversely isotropic plates are of the same type allows us to apply the results in Parts 1 and 2 to the case of three-layer plates without restrictions. Table 2 also shows the boundary conditions of the truncated problem.

It was proven that the Kirchhoff transform (generalized shearing force) exists in the neighborhood of a corner point. The notation ** in conditions 5 and 7 indicates that $(1-\vartheta)H_{0ns}$ should be replaced by M_{0ns} in the Kirchhoff transform, since M_{0ns} is the moment of the more general order. This substitution is made necessary by the fact that the asymptotic calculations are more rigorous than the initial hypotheses. The latter are based on the Grigolyuk–Chulkov theory of three-layer plates: the bearing layers are described by the Kirchhoff–Love theory, while the filler is described by a theory which allows for shear. Thus, the moment H_{ns} (describing the turning moment in the filler) needs to be replaced by M_{ns} (the turning moment for the entire plate).

The refined theories of Ambartsumyan for anisotropic plates and Grigolyuk and Chulkov for three-layer plates are of the same order of accuracy. Thus, the appearance of the additional terms in boundary conditions 1 and 5 in Table 2

$$(M_{0ns}|_{\Gamma_l \cap \Gamma_{l+1}}) f_{lk}(s), \quad |f_{lk}(s)| \leqslant C \exp(-\delta s)$$

has the same physical explanation as for the transversely isotropic plate. The presence of these terms is connected mainly with the accuracy of the hypotheses serving as the basis of the theory of bending for both transversely isotropic and three-layer plates. Only the use of more accurate hypotheses could yield a definitive answer to questions regarding the behavior of the solution in the neighborhood of corner points in the case when the bending and turning moments on the boundary are assigned.

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